INVESTMENT DECISIONS WITH BIRTHS AND DEATHS[1]

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Most Real Options authors have been assuming that the only critical variable in the investment decision analysis is price. The success of an investment depends not only on the price of a product but also on other variables like the quantity of production. The number of units sold is a variable with different characteristics than price. It can be in some applications a discrete variable, and its drift and volatility can be affected by different factors.
We present a model in which a monopoly investor has the option to invest in a new market, in which the number of units sold follows a stochastic birth and death process. We present a numerical solution for the value of the option to invest and for the trigger level for investment. Also we study the sensitivity of the option value to changes in the number of units sold and also the sensitivity of the trigger level to changes in volatility. The model suggests, that the classical assumption of Geometric Brownian motion, as the stochastic process of the underlying variable can overestimate the option value to invest.

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REAL OPTIONS WITH BIRTH AND DEATH PROCESSES

1- Introduction

In real options most authors use the geometric Brownian motion (GBM) to describe the underlying value, or profits, of the investment under analysis[3,4]. In order to model that the profit follows a geometric Brownian motion, authors normally assume that the costs and the number of units sold are constant, and since a GBM can explain the path followed by the price, the same process can, with these constraints, explain the path followed by the profit. Implicitly, authors have been assuming that the only critical variable in the investment decision analysis is price.

The success of an investment depends not only on the price of a product but also on other variables like the quantity of production. The number of units sold is a variable with different characteristics than price. It can be in some applications a discrete variable, and its drift and volatility can be affected by different factors. Birth and death processes have been used in disciplines like biology to explain the evolution of populations. Consequently, this seems a relevant way to describe the evolution of the number of units sold in a market composed of a population of active costumers.

We present a model in which a monopoly investor has the option to invest in a new market, in which the number of units sold follows a stochastic birth and death process. We present a numerical solution for the value of the option to invest and for the trigger level for investment. Also we study the sensitivity of the option value to changes in the number of units sold and also the sensitivity of the trigger level to changes in volatility.

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[3] To assume that the profit of a certain project can be described through a geometric Brownian motion is an unrealistic assumption. A geometric Brownian motion has, by definition, an absorbing barrier at zero (although zero is an unobtainable barrier). Profits can, and sometimes do, take negative values. Therefore the process is most of the time used for mathematical convenience.

[4] There are some exceptions to this rule. Some authors allow for jumps or mean reverting processes. Some examples are Gibson and Schwartz (1990), Brennan (1991), Cortazar and Schwartz (1994), Bessembinder et al. (1995), Bjerksund and Ekern (1995), Ross (1997), and Schwartz (1997).

2 - Birth and Death Processes

Birth and death processes are a kind of Markov process in discrete levels, which are often used to model the growth of populations. The population size is allowed to fluctuate instead of only increasing, as in the simple birth process, or only decreasing, as in the simple death process (Cox and Miller (1965); Karlin and Taylor (1975); Grimmett and Stirzaker (1992); Kao (1997); Taylor and Karlin (1998); Ross (2000)).

A birth and death process is an example of a discrete event process which develops in continuous time. In a birth and death process the discrete variable $X(t)$ symbolizes the size of a population at time *t* and $X(u)$ symbolizes the size of the population at a previous time *u*. The probability $P\left(X(t)=x \middle| X(u)=i\right)$ is normally called the transition probability, and in general depends on *i*, *x*, *u* and *t*. The probability $P\left(X(t)=x \middle| X(u)=i\right)$ will be denoted as $p_{i,x}(u,t)$ and should be read as the probability of X being in state *x* at time *t* knowing that *X* was in state *i* at a previous time *u*. For many Markov processes the time dependence of the transition probabilities relates only to the length of the interval *(t-u)*, so that for any *u*, *t* such that $0<u<t$, $p_{i,x}(u,t)=p_{i,x}(0,t-u)$ for all states *i*, *x*. In this case the Markov process is said to be homogeneous. In this paper we will present mainly a homogeneous birth and death process. The only exception will be introduced when we present a continuous time, continuous space birth and death process.

In any small interval of length δt, the probability that more than one event of either kind occurs is negligible; this means that in each small interval either a birth or a death will occur, or nothing will happen, since within a given interval two changes are impossible to second order. So, if $X(t)=x$ then the possible changes in the interval $[t, t+\delta t]$ are a birth, increasing the population size to $x+1$; or a death, reducing the size to $x-1$ at time $t+\delta t$. The probability that an event will occur, i.e. a birth or a death respectively, is given by $p_{x,x+1}$, and $p_{x,x-1}$. Since it is a Markov process, these probabilities depend only on *x*, *t* and δt, meaning that we may write:

$$p_{x,x+1}(t, t+\delta t) = \varphi_x \delta t + o(\delta t) \tag{1}$$

$$p_{x,x-1}(t, t+\delta t) = \upsilon_x \delta t + o(\delta t) \tag{2}$$

where φ_x and υ_x are functions of x and t (we will assume in most of our exposition that φ_x and υ_x are independent of time, meaning that the process is time homogeneous), and $o(\delta t)$ represents a negligible remainder term, in the sense that if we divide the term by δt, then the resulting value tends to zero as δt tends to zero. The variable φ_x is the overall birth rate and υ_x the overall death rate. Since the population size must be a non-negative integer, the death rate at moment zero is defined as zero, and the probability formulas hold even starting with zero individuals.

The probability that two or more events occur in any small interval is given by $o(\delta t)$ and the probability that no change occurs in any small interval is given by:

$$p_{x,x}(t, t+\delta t) = 1 - \varphi_x \delta t - \upsilon_x \delta t + o(\delta t) \tag{3}$$

Birth and death processes have mainly been applied in physics and biology but they can also describe phenomena like population growth, queuing models, and the number of clients using a certain product.

2.2 - Derivation of Birth and Death Processes

Suppose that we want to know the probability of having x individuals at time $t+\delta t$, keeping in mind that the probability of more than one event occurring is negligible over a small enough increment of time. At time $t+\delta t$, x individuals can be alive if one of the following things happen: at time t, there were $x-1$ individuals and one birth occurred; at time t, there were $x+1$ individuals and one death occurred; at time t, there were x individuals and nothing occurred.

The probabilities of occurrence of each event, already presented in the previous section, can also be presented as[5,6]:

$$p_{x-1,x}(t,t+\delta t)=\varphi_{x-1}\delta t+o(\delta t);\quad p_{x+1,x}(t,t+\delta t)=\nu_{x+1}\delta t+o(\delta t);$$
$$p_{x,x}(t,t+\delta t)=1-\varphi_x\delta t-\nu_x\delta t+o(\delta t)$$

The Chapman-Kolmogorov equation is normally used to derive the so called forward and backward equations. The Chapman-Kolmogorov equation is given by: $p_{i,x}(t_1,t_3)=\sum_{k=0}^{\infty}p_{i,k}(t_1,t_2)p_{k,x}(t_2,t_3)$. This equation holds for all times $t_1<t_2<t_3$ and for all states *i*, *k*, *x*. Using the Chapman-Kolmogorov equation the probability that at time $t+\delta t$ we have *x* individuals alive, knowing that at time zero there were *i* individuals alive, and that the process is time homogenous i.e. the transition probabilities are the same for all time intervals with the same length, the Chapman-Kolmogorov equation can be written as: $p_{i,x}(t+\delta t)=\sum_{k=0}^{\infty}p_{i,k}(t)p_{k,x}(t,t+\delta t)$. Consequently, $p_{i,x}(t+\delta t)$, becomes[7,8]:

$$\begin{aligned}p_{i,x}(t+\delta t)=&\left(\varphi_{x-1}\delta t+o(\delta t)\right)p_{i,x-1}(t)+\left(\nu_{x+1}\delta t+o(\delta t)\right)p_{i,x+1}(t)+\\&\left[1-\varphi_x\delta t-\nu_x\delta t+o(\delta t)\right]p_{i,x}(t)\end{aligned}\tag{4}$$

rearranging the terms:

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[5] Note that in the previous section the transition probabilities were presented as $p_{x,x-1}$ and not as $p_{x-1,x}$. The probability $p_{x-1,x}$ is the transition probability of the situation where the population size is *x-1* at time t and one birth occurs during δt. The probability $p_{x,x-1}$ is the transition probability of the situation where the population size is *x* at time *t* and one death occurs during δt.

[6] See group team of Open University Unit 8 (1988) for a similar derivation of birth and death processes.

[7] The Chapman Kolmogorov equation in continuous time covers the process in three time points $t_1< t_2< t_3$. In the present case $t_1=0;\ t_2=t$ *and* $t_3=t+\delta t$. We will omit zero from the notation. Therefore we will write $p_{i,x}(t+\delta t)$ instead of $p_{i,x}(0,t+\delta t)$.

[8] The only values of *k* whose probabilities are non negligible are: $k=x-1$; $k=x+1$; $k=x$.

$$\frac{p_{i,x}(t+\delta t)-p_{i,x}(t)}{\delta t}=\left(\varphi_{x-1}+\frac{o(\delta t)}{\delta t}\right)p_{i,x-1}(t)+\left(\nu_{x+1}+\frac{o(\delta t)}{\delta t}\right)p_{i,x+1}(t)$$
$$-p_{i,x}(t)\left(\varphi_x+\nu_x+\frac{o(\delta t)}{\delta t}\right)$$

Postulating that at moment zero the death rate is equal to zero, the previous equation will hold even when there are no individuals alive at time zero. Letting $\delta t \to 0$ the Kolmogorov forward equation for the birth and death process is as follows:

$$\frac{dp_{i,x}(t)}{dt}=\varphi_{x-1}p_{i,x-1}(t)+\nu_{x+1}p_{i,x+1}(t)-(\varphi_x+\nu_x)p_{i,x}(t) \tag{5}$$

The forward equation results from splitting the interval $(0,t+\delta t)$ into $(0,t)$ and $(t,t+\delta t)$. Another result can be obtained when the first step of examining the transition probability is the short time interval $-\delta t$. Consequently, the time interval is now separated into $(-\delta t,0)$ and $(0,t)$ and, similarly to what was done above, the transition probability is examined in each period separately (notice that the length of the interval is still $t+\delta t$). This results in the backward equation:

$$\frac{dp_{i,x}(t)}{dt}=\varphi_i p_{i+1,x}(t)+\nu_i p_{i-1,x}(t)-(\varphi_i+\nu_i)p_{i,x}(t) \tag{6}$$

The forward equation will tend to be used if there is a single initial state of particular importance and we want to know the probabilities at time *t* for various final states. Conversely, the backward equation is normally used if there is a single final state of great interest and we want the probability of reaching this state at time *t* for various initial states (Cox and Miller, 1965).

2.2.1 - Simple Birth and Death

In the simple birth and death process, it is assumed that each individual, independently of all other individuals, gives birth to new individuals, one at a time, at rate φ. Each individual is liable to die, with the lifetime of each individual having an exponential distribution with parameter ν.

The distribution of the random variable of total population size can be derived, recursively, using the differential-difference equations or the probability generation function. The differential-difference equations, express the probability $p_{i,x}(t)$ in terms of $p_{i,x-1}(t)$ and $p_{i,x+1}(t)$. For example, for the birth and death process the differential-difference equation are given by[9]:

$$\frac{dp_{i,x}(t)}{dt} = \varphi_{x-1} p_{i,x-1}(t) + \nu_{x+1} p_{i,x+1}(t) - (\varphi_x + \nu_x) p_{i,x}(t) \quad \text{for x=1,2,...} \qquad (7\text{-a})$$

$$\frac{dp_{i,0}(t)}{dt} = \nu_1 p_{i,1}(t) + \varphi_0 p_{i,0}(t) \qquad \text{for x=0} \qquad (7\text{-b})$$

Now we will calculate the probability generating function of the total population size. The probability generating function (denoted by $\Omega(s,t)$) contains full information on the probability distribution of *X(t)* (the size of the population at time t). Probability generating functions are used mainly to calculate moments and to calculate the distribution of sums of independent random variables. The probability generating function is given by $\Omega(s,t) = \sum_{x=0}^{\infty} p_{i,x}(t) s^x$ (for an $s \in \Re$ for which the sum converges)

In order to calculate the probability generating function of the state variable *X(t)*, the differential-difference equations of the simple birth and death, are obtained substituting $\varphi_x \to \varphi x$ and $\nu_x \to \nu x$ in the Kolmogorov forward equation[10]:

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[9] Notice that at time zero the death rate is zero.

[10] Since all individuals act independently, if *X(t)=x* at time *t*, the probability of one individual being born in the interval $[t, t+\delta t]$ is given by $\varphi x \delta t + o(\delta t)$. On the other hand, since each

$$\frac{dp_{i,x}(t)}{dt} = \varphi(x-1)p_{i,x-1}(t) + \nu(x+1)p_{i,x+1}(t) - (\varphi x + \nu x)p_{i,x}(t) \quad \text{for x=1,2…} \qquad (8\text{-a})$$

$$\frac{dp_{i,0}(t)}{dt} = \nu p_{i,1}(t) \qquad \text{for x=0} \qquad (8\text{-b})$$

So far, we have assumed that changes in population state are always of dimension one (and that the transition rates were not dependent on time). In other words, the population size evolves by increasing or decreasing by one individual. Let the value of the state variable at an initial time, u, be denoted by i, i.e. ($X(u) = i$), and the value of the state variable at a later time t be denoted by x, i.e. ($X(t) = x$). We will now assume that the birth and death rates are dependent on time. A natural consequence of this is that the process is not time homogeneous and therefore $p_{i,x}(u,t)$ can no longer be written as $p_{i,x}(0,t-u)$ or $p_{i,x}(t-u)$ suppressing zero. Consequently, the backward equation of the general birth and death process (equation (6)) is now given by[11]:

$$-\frac{\partial p_{i,x}(u,t)}{\partial u} = \varphi_i(u)p_{i+1,x}(u,t) + \nu_i(u)p_{i-1,x}(u,t) - (\varphi_i(u) + \nu_i(u))p_{i,x}(u,t) \qquad (9)$$

We will also assume that the changes in state can be of dimension[12] δi. Consequently, the possible changes in the interval δt are a birth, increasing the population size to $i + \delta i$, or a death, reducing the size to $i - \delta i$. In this specification the population size is analysed as the position of a continuous random variable, but its dynamics are different from those of the GBM.

individual is liable to die, independently of giving birth, the probability that one individual dies in the time interval $[t, t+\delta t]$ is given by $\nu x \delta t + o(\delta t)$.

[11] See Cox and Miller (1965) page 181 for details.

[12] For details on the diffusion limit of a simple birth and death process see Cox and Ross (1976).

According to Cox and Ross (1976) the backward equation for a continuous time continuous state simple birth and death process is given by[13]:

$$-\frac{\partial p_{i,x}(u,t)}{\partial u} = i\varphi(u)p_{i+\delta i,x}(u,t) + i\nu(u)p_{i-\delta i,x}(u,t) - i(\varphi(u)+\nu(u))p_{i,x}(u,t) \quad (10)$$

If at time u the process is at i, then in the next small time interval the change in position will be $-\delta i$, 0 or δi with probabilities $\nu(u)i$, $(1-\varphi(u)i)-\nu(u)i)$ and $\varphi(u)i$. Consequently, the instantaneous mean change in position, $\mu(u)$ is[14]:

$$\mu(u) = (\varphi(u)-\nu(u))\delta i \quad (11)$$

and the variance, $Var(u)$, of the change in position:

$$Var(u) = (\varphi(u)+\nu(u))\delta i^2 \quad (12)$$

According to Cox and Ross (1976) if we now let δi approach zero in (10) maintaining the mean and variance of the process as defined above, we obtain the backward equation for a continuous state simple birth and death, or in other words the backward equation for the continuous diffusion of a simple birth and death process[15,16]:

$$\begin{aligned} -\frac{\partial p_{i,x}(u,t)}{\partial u} &= -i(\varphi(u)+\nu(u))p_{i,x}(u,t) + \\ & i\varphi(u)\left[p_{i,x}(u,t) + \frac{\partial p_{i,x}(u,t)}{\partial i}\delta i + \frac{1}{2}\frac{\partial^2 p_{i,x}(u,t)}{\partial i^2}\delta i^2\right] \\ & + i\nu(u)\left[p_{i,x}(u,t) - \frac{\partial p_{i,x}(u,t)}{\partial i}\delta i + \frac{1}{2}\frac{\partial^2 p_{i,x}(u,t)}{\partial i^2}\delta i^2\right] \end{aligned} \quad (13)$$

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[13] Notice that if in equation (9) we substitute $\nu_i(u) = i\nu(u)$ and $\varphi_i(u) = i\varphi(u)$ (the rates of the simple birth and death) and in (9) we let the increases/decreases in state be of length δi and not 1, we obtain equation (10). It is required that the derivatives exist at the origin.

[14] See page 214 of Cox and Miller (1965) for details.

[15] It was assumed that our function can be differentiated a number of times and expanded in a Taylor series.

[16] For a rigorous derivation of the diffusion birth and death see Feller (1959).

rearranging (13):

$$-\frac{\partial p_{i,x}(u,t)}{\partial u}=(\varphi(u)-\nu(u))i\frac{\partial p_{i,x}(u,t)}{\partial i}\delta i+(\varphi(u)+\nu(u))i\frac{1}{2}\frac{\partial^2 p_{i,x}(u,t)}{\partial i^2}\delta i^2$$

or using (11) and (12):

$$-\frac{\partial p_{i,x}(u,t)}{\partial u}=\mu(u)i\frac{\partial p_{i,x}(u,t)}{\partial i}+Var(u)i\frac{1}{2}\frac{\partial^2 p_{i,x}(u,t)}{\partial i^2} \tag{14}$$

Equation (14) is the backward equation for the continuous diffusion of a simple birth and death process.

According to Cox and Ross (1976), the stochastic differential equation corresponding to the backward diffusion equation (14) is given by (where $\mu(u)$ and $Var(u)$ are assumed to be constant through time, as it is common in for example the Brownian motion processes, i.e. μ and Var respectively):

$$dX(t)=\mu X(t)dt+\sqrt{X(t)Var}dz \tag{15}$$

where *dz* is the increment of a Wiener process[17]. In equation (15) *dX* represents the change in the population size in a small instant of time δt. This change is

[17] The stochastic differential equation of the simple birth and death process (15) can be obtained comparing the backward equation of the simple birth and death process (14), with the backward equation of the geometric Brownian motion. Notice that the corresponding stochastic differential equation of the geometric Brownian motion is: $dX(t)=\mu_g X(t)dt+\sqrt{Var_g}X(t)dz'$ where μ_g and Var_g denote respectively the drift and the variance of the process. The backward equation if *X(t)* follows a geometric Brownian motion is: $-\frac{\partial p_{i,x}(t_0,t)}{\partial t_0}=\mu_g i\frac{\partial p_{i,x}(t_0,t)}{\partial i}+Var_g i^2\frac{1}{2}\frac{\partial^2 p_{i,x}(t_0,t)}{\partial i^2}$ where μ_g and Var_g are constant through time (see page 144 of Wilmott (1998)). Notice that the difference between the former equation and our equation (14) is not only the drift and the variance, but also the fact that the second term in the right hand side is multiplied by i^2 and not, as in (14), by *i*. The stochastic differential equation of the geometric Brownian motion, the first equation in this footnote, has the Wiener term *dz'* multiplied by the standard deviation of the process and the state variable. In contrast, the stochastic differential equation of the birth and death, (15), has the Wiener term *dz* multiplied by the standard deviation of the process and the square root of the state variable.

explained by two elements, a drift term μdt corresponding to a growth rate which is certain, and a normally distributed stochastic term $\sqrt{Var}dz$.

3 – Number of Units as a Birth and Death Process - The Monopolist's Option to Invest

We will now derive the option value function of a risk neutral monopoly investor. We will also assume that the profits per unit equal one and that one unit per annum is sold to each customer, the number of units sold, denoted by M_t, follows a stochastic simple birth and death process. Using the stochastic differential equation (15) defined for the simple birth and death we can write the stochastic differential diffusion equation for M_t:

$$dM_t = \omega M_t dt + \alpha\sqrt{M_t}dz . \tag{16}$$

where ω is the drift and α the standard deviation of the process[18]. The mean and variance of this new process are as defined by equations (11) and (12) and assumed to be constant though time.

The stochastic process described by equation (16) is significantly different from the usual geometric Brownian motion. In the birth and death process, each unit sold is stochastically independent of the others, i.e. a customer[19] can enter independently of another, and he can leave at any time. If the number of units were explained by a geometric Brownian motion the events would be perfectly dependent on each other i.e. when one customer moves they all move. Another difference lies in the variance. The variance of a geometric Brownian motion is given by $\alpha^2 M^2$, but for a birth and death process it is given by $\alpha^2 M$. Cox and Ross (1976) present the diffusion limit of the birth and death process as accurate to describe situations in which changes in state are small, as in the geometric Brownian motion, but in which the variance of

[18] We will omit the time subscripts from now on.

[19] Assuming, without any loss of generality, that one customer purchases one unit.

the variable, in this case the number of units sold, increases with the state variable, although more slowly than if the process was Brownian.

Let $P_0(M)$ denote the value of a perpetual option to invest in a monopoly market where the number of units, *M,* follows a stochastic birth and death process. We will construct a portfolio, which does not pay dividends and therefore, its return will only be capital gains. Such a portfolio will be formed by a long position in the option and a short position in Δ units of *M*. Any change in this portfolio can be explained by:

$$dP_0(M) - \Delta dM - (r-\omega)M\Delta dt \tag{17}$$

where $(r-\omega)M\Delta dt$ represents the rate of dividend which M requests on its own capital value. Expanding dP_0 using Ito's Lemma:

$$dP_0(M) = \frac{dP_0(M)}{dM}dM + \frac{1}{2}\frac{d^2P_0(M)}{dM^2}dM^2 \tag{18}$$

Substituting (16) and (18) into (17), recollecting the terms and using $dM^2 = \alpha^2 M dt$, we obtain:

$$\frac{1}{2}\alpha^2 M \frac{d^2P_0(M)}{dM^2}dt + \omega M \frac{dP_0(M)}{dM}dt - \omega M\Delta dt - (r-\omega)M\Delta dt +$$
$$\alpha\sqrt{M}\frac{dP_0(M)}{dM}dz - \alpha\sqrt{M}\Delta dz$$

If $\Delta = \dfrac{dP_0(M)}{dM}$ the two random terms and the two trend terms in the previous equation disappear yielding:

$$\frac{1}{2}\alpha^2 M \frac{d^2P_0(M)}{dM^2}dt - (r-\omega)M \frac{dP_0(M)}{dM}dt \tag{19}$$

Since there is no randomness in this portfolio, its return should be a risk free return. Thus:

$$\frac{1}{2}\alpha^2 M \frac{d^2P_0(M)}{dM^2}dt-(r-\omega)M\frac{dP_0(M)}{dM}dt=r\left[P_0(M)-\frac{dP_0(M)}{dM}M\right]dt \qquad (20)$$

Dividing (20) by dt and rearranging, we obtain the differential equation that explains the movements in the value $P_0(M)$ of the opportunity to invest:

$$\frac{1}{2}\alpha^2 M \frac{d^2P_0(M)}{dM^2}+\omega M\frac{dP_0(M)}{dM}-rP_0(M)=0 \qquad (21\text{-a})$$

We will now solve this differential equation to determine the trigger value of M at which the monopolist will exercise his perpetual option to invest, denoted by M^*. We will determine M^* as a function of the volatility α, treating the variables r and ω as known quantities.

3.1 – Numerical Solution

Since (21-a) has no closed form solution, an innovative numerical solution is proposed. There is a problem with direct integration. The equation is singular at $M=0$ (the highest derivative is multiplied by M) and simple integration methods fail. Notice that (21-a) can be presented as:

$$\frac{d^2P_0(M)}{dM^2}+2\frac{\omega}{\alpha^2}\frac{dP_0(M)}{dM}-\frac{2r}{\alpha^2 M}P_0(M)=0$$

At $M=0$ the term $-\frac{2rP_0(M)}{\alpha^2 M}$ fails to be analytical, consequently there is no solution at that point. Thus, it is necessary to construct an analytical solution for use near $M=0$. This can be used to calculate the values of $P_0(M)$ and its derivative at a point near to $M=0$ (but not at $M=0$ itself). A numerical routine can then be employed to extend the solution to higher values of M. We now derive a solution method, which seems new to the existing real options literature.

We begin by introducing the scaled variable ξ, defined by $M = \alpha^2 \xi$, in place of M. Then equation (21-a) can be written as:

$$\frac{1}{2}\frac{d^2P(\xi)}{d\xi^2}\xi + \omega\xi\frac{dP(\xi)}{d\xi} - rP(\xi) = 0 \qquad \text{(21-b)}$$

Notice that this form of the differential equation does not explicitly contain the volatility parameter α. Thus we can construct universal solutions of this equation that apply regardless of the value of α. This is true for both the numerical method and the analytical solution around $M=0$. The parameter α is still present, however, since it appears in the boundary conditions. Having determined the universal solution, we then apply these boundary conditions and thus determine how the trigger M^* depends on the value of α.

Equation (21-b) does not have any explicit solution. As we have said we need an analytic solution for application near $M=0$. We can find one in the form of a Frobenius series[20], namely (where c represents the roots of the indicial equation defined below):

$$P_0(\xi) = \sum_{n=0}^{\infty} a_n \xi^{n+c}$$

The convergence of the power series $P_0(\xi)$ is given by the limit of the radius of convergence ρ, where the radius of convergence is defined as: $\rho = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. If $\rho=0$, then the series diverges for all $\xi \neq 0$; if $0 < \rho < \infty$, then the series converges if $|\xi| < \rho$ and diverges if $|\xi| > \rho$; if $\rho = \infty$, then the series converges for all ξ. Even if the limit defined above does not exist, there will

[20] Some of the features of the solutions of differential equations of most importance for applications are determined near their singular points. (See Edwards and Penney,1985, for details).

always be a number ρ such that one of the three alternatives defined above holds.

To make this series specific to the problem at hand, the Frobenius series is substituted into (21-b) to obtain, after rearrangement:

$$\sum_{n=-1}^{\infty}\frac{1}{2}a_{n+1}(n+1+c)(n+c)\xi^{n+c}+\sum_{n=0}^{\infty}\left[\omega(n+c)-r\right]a_n\xi^{n+c}=0 \tag{22}$$

We now compare the coefficients of the different powers of ξ. If *n=-1* (first term only) we obtain the indicial equation[21]:

$$\frac{1}{2}a_0c(c-1)=0$$

Since $a_0 \neq 0$ (the first term must exist), this determines the possible values of the index *c*. According to the Frobenius method, equation (21-b) has two solutions. The first solution corresponds to the larger root of the indicial equation. In other words, the first solution is obtained by substituting the larger root of *c* in the Frobenius series. The second solution exists only if the difference between the two roots is neither zero, nor a positive integer. Here there are two roots, *c=0* and *c=1*.Therefore the indicial equation for the lower power of ξ is a failing Frobenius case, meaning that the smaller root of this equation does not lead to an acceptable approximation of the value of the differential equation, (21-b) near the value $\xi = 0$. This is because the two roots differ by an integer[22]. When the roots differ by an integer, the Frobenius method guarantees the existence of the solution in the form of a Frobenius series, for the highest root only, in this case *c=1*. If the two roots are equal, there is only one solution, but this is not true in the present case so a second solution must be found. In general, the root *c=1* gives a solution of the Frobenius kind, and the second solution for the root *c=0* will be a Frobenius

[21] The term a_{-1} does not exist by definition.

[22] See Edwards and Penney (1985) for a detailed explanation of the Frobenius method.

series for *c=0* plus the logarithm of the solution[23] for *c=1*. Therefore, in such cases a first independent solution can be constructed based on the *c=1* root. A second independent solution can be constructed from a series solution based on *c=0*, plus the term $\ln(\xi) * (\text{first solution})$. That is (where $P_0 1$ and $P_0 2$ denote the first and second solution)[24]:

$$P_0 1(\xi) = \sum_{n=0}^{\infty} a_n \xi^{n+1}\ , \quad \text{and} \qquad\qquad P_0 2 = \ln(\xi) P_0 1(\xi) + \sum_{n=0}^{\infty} \iota_n \xi^n \tag{23}$$

We now impose a boundary condition on P_0 to specify the solution more closely.

As the state variable *M* goes to zero (and hence $\xi \to 0$), the value of the option to invest has necessarily to decrease. When the state variable reaches zero, the option to invest is worthless (so $P_0(\xi)$ must equal 0)[25]. This is automatically satisfied by the $P_0 1(\xi)$ function (notice that if $\xi = 0$ all the terms of $P_0 1(\xi)$ will also equal zero), but the $P_0 2(\xi)$ contains a constant term ι_0 (Notice that the first term of $P_0 2(\xi)$, i.e. when *n=0*, is undefined if $\xi = 0$). Therefore, to satisfy this boundary condition the $P_0 2(\xi)$ term must be absent from the solution and the value $P_0(\xi)$ must be a multiple of $P_0 1(\xi)$ only.

We now find the solution $P_0 1(\xi)$. There is an arbitrary constant in this solution, the value of a_0, which we call *A* (looking at equation (25) below, we can see that all the terms can be presented as a product by a_0). Since the coefficients are linear, all higher coefficients of the solution will have a factor of *A*. Rather than carry this factor in our working, we will calculate the particular function $P_h(\xi)$ that has a_0=1, i.e. $P_h(\xi) = P_0 1(\xi)\big|_{a_0=1}$. The general solution will then take the form $P_0(\xi) = A P_h(\xi)$.

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[23] In our case the solution when *c=1*.

[24] Where ι_n denotes the coefficients of the series when *c=0*.

[25] Notice that the birth and death rates, in simple birth and death processes, are linear in the size of the population. For example the birth rate is given by φx, therefore zero is an absorbing state: if the size of the population is ever zero, it will stay zero forever. This will not be necessarily true if immigration is allowed.

The remaining terms of the power series are obtained by comparing the coefficients of the different powers of ξ in (22) for $n \geq 0$ and *c=1*. We obtain:

$$\frac{1}{2} a_{n+1}(n+2)(n+1) + [\omega(n+1) - r]a_n = 0, \qquad n \geq 0 \tag{24}$$

Solving (24) and changing the counter from *n+1* to *n*, we obtain:

$$a_n = \frac{2[r - n\omega]}{(n+1)n} a_{n-1}, \qquad n \geq 1 \tag{25}$$

This equation can be used to obtain an approximate solution by determining iteratively as many of the coefficients in the power series of the solution as we desire (from a_0 we can find a_1, then from a_1 we can find a_2, etc)[26]. We will do this in a numerical programme later.

Besides the boundary conditions at $M(=\xi) = 0$ the ODE is subject to the usual two additional boundary conditions namely value matching (notice that both the notional price per unit and the notional annual rate of units sold, for any of the *M* "costumers" served, are assumed to be unity, and therefore *Mdt* represents the cash flow received during *dt*):

$$P(M_F) = \frac{M_F}{r - \omega} - K \qquad \text{or} \qquad P(\xi_F) = \frac{\alpha^2 \xi_F}{r - \omega} - K \tag{26}$$

and smooth pasting:

$$\frac{dP(M_F)}{dM} = \frac{1}{r - \omega} \qquad \text{or} \qquad \frac{dP(\xi_F)}{d\xi} = \frac{\alpha^2}{r - \omega} \tag{27}$$

[26] Notice that the radius of convergence of our power series is given by: $\rho = \lim_{n\to\infty} \left| \frac{(n+2)(n+1)}{2[r-(n+1)\omega]} \right| = \infty$. Consequently, the series converges for all ξ.

where after each boundary condition on M we have given the equivalent boundary condition for the scaled variable ξ.

After designing a solution near the origin, our final stage is to design a computer programme that determines the solution $P_0(\xi) = AP_h(\xi)$ subject to the two boundary conditions and the value of ξ_F that satisfies the two boundary conditions, which is the trigger value at which the monopolist invests.

The computer programme begins by calculating $P_h(\xi)$. As described above this is done by using the Frobenius power series to calculate the function at a point near to $\xi = 0$. The programme then uses a numerical integration routine (the Runge-Kutta[27] scheme) to continue the integration further. In the sample programme we have (arbitrarily) integrated numerically as far as $\xi = 300$.

We now use the numerical solution to calculate the following quantities for all values of ξ on the integration range[28]:

$$A_1 = \frac{1}{P_h(\xi)}\left[\frac{\alpha^2 \xi}{r-\omega} - K\right] \qquad\qquad A_2 = \frac{1}{dP_h(\xi)/d\xi}\left[\frac{\alpha^2}{r-\omega}\right]$$

At $\xi = \xi^*$ both these functions should have the same value (since then $A=A_1=A_2$, and (26) and (27) are satisfied). Consequently the final part of the solution is determined by comparing A_1 with A_2 for values of ξ increasing from zero. At the value of ξ where $A_1 - A_2$ changes sign we know that ξ^* lies between that value of ξ and the previous one, so we use linear interpolation between the two to estimate the actual value of ξ^*.

The evaluation of ξ^* is placed in a loop that does the calculation over a range of values for α (and a given fixed value for K). These values are recorded. From ξ^*, M^* can be calculated so we can plot the final graph of M^* against α.

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[27] See e.g. Edwards and Penney (1985) for a detailed explanation of the Runge-Kutta method.

[28] Notice that until now we have the solution of $P_h(\xi)$. P_0 is found by multiplying $P_h(\xi)$ by the arbitrary constant A. This is also valid for the boundary conditions (26) and (27) where $P(\xi^*)$ and its derivative should be substituted by $P_h(\xi)$ and its derivative multiplied respectively by the arbitrary constants A_1 and A_2.

3.2 – Numerical Results

In Figure 1, we present the value of the monopolist's opportunity to invest and the net present value as a function of the number of units M which can be sold in the market place.

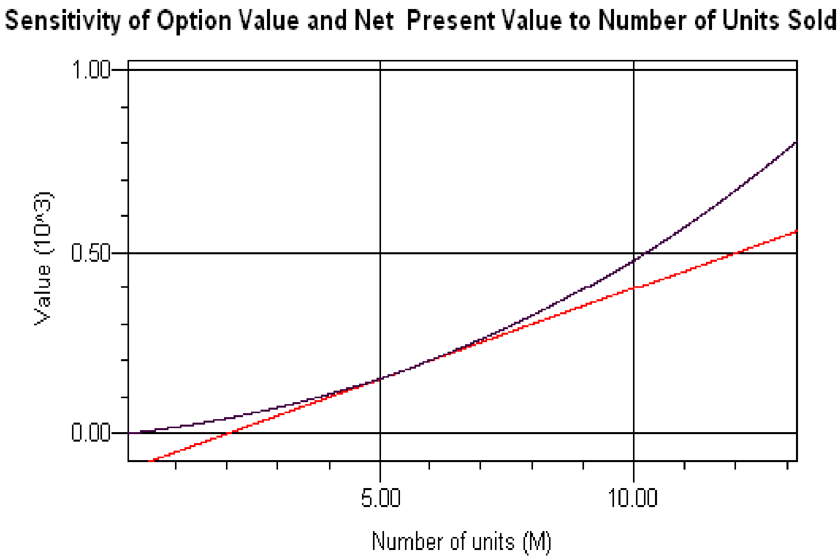

Fig. 1 – The parameters are: $r=3\%$, $\alpha=25\%$, $\omega=1\%$ and $K=100$

The top line in Figure 1 represents the option value to invest. The bottom line represents the net present value. We can see that the two lines meet when the number of units sold M equals 5.52. Consequently for number of units sold, M, smaller than 5.52, the monopolist will be idle, and the value of his opportunity to invest is given by the option to invest (the top line). If the number of units sold, M, is larger than 5.52 the value of the investment is given by the net present value (bottom line). We can also see from the graph that as the number of units sold increases both the value of the option to invest and the net present value increase, in a monopoly market where the number of units evolves according to a birth and death process.

For comparison purposes we estimated the trigger, M^*, using the model of Dixit and Pindyck (1994) in which the underlying variable follows a geometric Brownian motion[29]. Using the above parameters, we obtain a trigger of 7.307. Therefore, the assumption that the underlying variable follows a geometric Brownian motion leads to higher option values, and consequently higher optimal investment times. This conclusion can be expected just by looking at the stochastic differential equation (16) of the number of units sold, M. Notice that in (16) the volatility is multiplied by $\sqrt{M}$, and not by M as in the stochastic differential equation of the geometric Brownian motion. Therefore, the volatility of M increases more slowly with M if the stochastic process is a Birth and death, than if it is Brownian. This effect necessarily leads to lower option values.

We reproduce below in Figure 2 the sensitivity of monopolist's trigger level M^* to volatility.

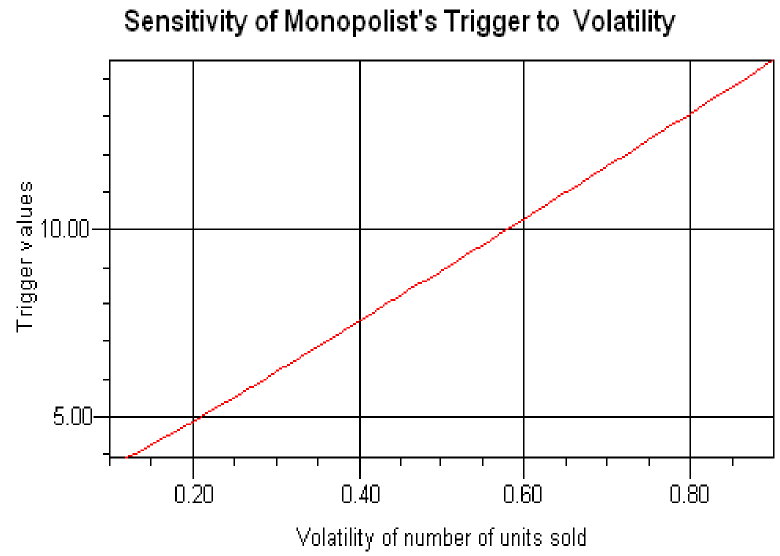

Fig. 2 – The parameters are: $r=3\%$, $\omega=1\%$ and $K=100$

As volatility increases, the trigger level M^* increases. Therefore in highly volatile markets, investors will defer the exercise of their rights to invest.

[29] The model presented in page 184 of Dixit and Pindyck (1994) determines the optimal time to invest and the option to invest, in a monopoly market where the price of the underlying follows a geometric Brownian motion. We are here using that model with the straightforward adaptation that the underlying variable is number of units sold, and not price.

4 - Conclusion

In this paper we have introduced a different dynamic model for the number of units sold, namely a birth and process.

Birth and death processes have been used extensively in disciplines like biology to describe the evolution of populations. Consequently, this seems a relevant way to describe the evolution of the number of units sold by a monopolist in a market composed of a population of active costumers. We have presented the birth and death process both as a continuous time, discrete state process, and also at its limit, as a diffusion process. Considering a market where the profits per unit equal one, the notional annual rate of units sold, for any of the "costumers" served, are assumed to be unity and the number of units follows a stochastic birth and death process, we derive the value of the monopoly right to invest in such a market and the decision rule for exercising that right. Not being able to find a closed form solution for the option to invest, a numerical solution is presented. The sensitivity of the value of the option to invest to changes in the number of units sold is as expected. As the number of units sold increases, the value of the option to invest in the market also increases. The sensitivity of the investment trigger level to volatility was also studied. The usual results were obtained, i.e. the investor will tend to delay his entry in more volatile markets.

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